Theory of selective excitation of multiple-quantum transitions

W. S. Warren, D. P. Weitekamp, and A. Pines

Department of Chemistry and Materials and Molecular Research Division, Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720

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The question of whether a molecule can be made to absorb and emit photons only in groups of \( n \) is treated. Pulse sequences are introduced which in effect selectively induce the absorption of only groups of \( n \) photons. This causes only \( n \)-quantum transitions even when many other transitions might be resonant. The technique involves repeated phase shifts of \( \pi n/2 \) in the radiation to build up the selected coherences and cancel all other coherences, and is applicable to a wide range of spectroscopic systems. Coherent averaging theory is extended to describe selective sequences and demonstrates that \( n \)-quantum selectivity is possible to arbitrarily high order in the average Hamiltonian expansion. High-order selectivity requires many phase shifts, however, and for this reason the residual nonselective effects of sequences which are selective to only a finite order are calculated. Selective sequences are applied to the multiple-quantum NMR of oriented molecules, where in combination with time reversal sequences they produce a much more efficient transfer of the population differences into selected coherences than is obtainable by normal wideband pumping. For example, the 10-quantum transition in a 4-spin system can be enhanced by more than four orders of magnitude. Experiments on selective excitation of the 4-quantum transitions in oriented benzene verify the expected enhancement.

I. INTRODUCTION

It has recently been shown that the Zeeman quantum number selection rule \( \Delta m = 1 \) of conventional Fourier transform NMR can be overcome in a systematic manner by suitably designed pulse sequences, thus permitting the observation of multiple-quantum (\( \Delta m = n \)) coherences.\(^1\)\(^-\)\(^4\) The usefulness of observing multiple-quantum transitions can be understood by considering a schematic energy level diagram for a system of \( N \) spins \( \frac{1}{2} \) without symmetry, shown in Fig. 1. The number of levels for each allowed eigenvalue of \( I_z = m \) from \( m = N/2 \) to \( m = -N/2 \) is \( \binom{N}{m} \). With this distribution, the number of allowed single-quantum transitions increases exponentially with \( N \), and in general the single-quantum transitions are only resolvable if there are few spins, or if many of the transitions are either degenerate or forbidden. However, there is only one transition with \( \Delta m = N \), corresponding to all \( N \) spins flipping together, and at most \( 2N \) transitions with \( \Delta m = N - 1 \). In an oriented system such as a liquid crystal, the total spectral width may be many kHz, with each line typically a few Hz wide. Thus, high multiple-quantum spectra are well resolved even if \( N \) is large.

This point is illustrated by Fig. 2, which shows the multiple-quantum spectra of benzene oriented in a liquid crystalline solvent. The six-quantum spectrum has only one line, and the five-quantum spectrum has only two lines because of the high molecular symmetry; the normal single-quantum spectrum has 76 lines.\(^5\) Unfortunately, the enhanced simplicity of multiple-quantum spectra has to be balanced against a reduced overall intensity. On the average, every symmetry allowed transition is pumped about equally well (independent of \( \Delta m \)) in the standard multiple-quantum experiment.\(^6\) As a result, total intensities of the high multiple-quantum spectra in these totally nonselective experiments are much weaker than the total intensity of the single-quantum spectrum, and most of the spectral intensity is "wasted" in unresolvable transitions.

Clearly, it would be extremely valuable if we were able to distribute the total spectral intensity between only a few orders of coherence, instead of driving all transitions equally. From the viewpoint of perturbation theory this appears impossible because a multiple-quantum transition occurs only with irradiation which also excites lower quantum transitions. As a result, total intensities of the high multiple-quantum spectra in these totally nonselective experiments are much weaker than the total intensity of the single-quantum spectrum, and most of the spectral intensity is "wasted" in unresolvable transitions.

FIG. 1. Energy level diagram for a general system with \( N \) spins \( \frac{1}{2} \). Each value of \( M \) has degeneracy \( \binom{N}{2M} \), corresponding to a binomial distribution.
transitions are simultaneously resonant as required by the nearly equal spin energy level spacing. Thus, aside from even-odd selection due to the bilinear form of spin coupling operators, no general method of selective excitation has been proposed. In this paper and the letter which preceded it, we demonstrate that selective excitation of multiple-quantum coherences in NMR is theoretically and experimentally possible as illustrated in Fig. 3. We will show that this technique can provide enormous signal enhancement, and that general selective sequences are applicable to a wide range of spectroscopic systems.

II. GENERAL THEORY OF SELECTIVE SEQUENCES

A. Review of average Hamiltonian theory

The effect of any sequence of irradiating pulses and delays on a general system in the absence of relaxation can be represented by a single unitary transformation $U$, called the propagator. Calculating $U$ directly by multiplying together the propagators for each part of the sequence is extremely tedious if many eigenstates are involved. However, this calculation can be avoided for certain pulse sequences by a technique known as average Hamiltonian theory. This technique is thorough-
ly documented, \(^8,^9\) so we will reproduce in this sub-
section only a brief summary of previous results which we
require for our development. In the next section we will
generalize the results to describe sequences which are
inherently selective.

The total Hamiltonian of a system is written as \(\mathcal{H}(t) = \mathcal{H}_{\text{int}} + \mathcal{H}_{\text{ext}}(t)\), where \(\mathcal{H}_{\text{int}}\) is the internal Hamiltonian of the system (for example, the interactions between pairs of 
magnetic dipoles) and \(\mathcal{H}_{\text{ext}}(t)\) is the explicitly time-de-
pendent interaction controlled by the experimenter (for example, the interaction with radiation). \(\mathcal{H}_{\text{ext}}(t)\) is termed
cyclic with cycle time \(t_c\) if \(\mathcal{H}_{\text{ext}}(t)\) and the propagator \(U(t) = T \exp \left[ -i \int_0^t \mathcal{H}_{\text{ext}}(t') dt' \right] \) (\(T\) is the Dyson time-ordering operator) are periodic, and if \(t_c\) is the shortest interval that constitutes a period for both \(U(t)\) and \(\mathcal{H}_{\text{ext}}(t)\).

\(\mathcal{H}_{\text{int}}\) is considered to be time independent, so \(\mathcal{H}(t)\) is cyclic if \(\mathcal{H}_{\text{ext}}(t)\) is cyclic. If \(\mathcal{H}_{\text{ext}}(t)\) is a pulse sequence made up of an integral number \(N\) of cycles, the propa-
gator for the entire sequence is the \(N\)th power of the
propagator corresponding to one cycle, and therefore only a single cycle need be considered.

The propagator for a single cycle can be shown to be
\[
U = \exp(-i\mathcal{H}t_c) = \exp \left[ -i(\mathcal{H}(0) + \mathcal{H}(1) + \ldots + \mathcal{H}(n)) \right] \tag{1}
\]
where
\[
\mathcal{H}(0) = \frac{1}{t_c} \int_0^{t_c} \mathcal{H}_{\text{int}}(t) dt ,
\]
\[
\mathcal{H}(1) = \frac{1}{2t_c} \int_0^{t_c} dt_1 \int_0^{t_c} dt_2 [\mathcal{H}_{\text{int}}(t_2), \mathcal{H}_{\text{ext}}(t_1)] ,
\]
\[
\mathcal{H}(2) = \frac{1}{8t_c^2} \int_0^{t_c} dt_1 \int_0^{t_c} dt_2 \int_0^{t_c} dt_3 \mathcal{H}_{\text{ext}}(t_3) \times [\mathcal{H}_{\text{int}}(t_2), [\mathcal{H}_{\text{int}}(t_1), [\mathcal{H}_{\text{int}}(t_2), \mathcal{H}_{\text{int}}(t_1)]] ,
\]
and
\[
\mathcal{H}_{\text{int}}(t) = U(t^{-1})(t)\mathcal{H}_{\text{int}} U(t) .
\]

This is merely a Magnus expansion\(^10\) of the propagator in
powers of the cycle time. The average Hamiltonian expansion is a perturbation expansion in powers of a smallness parameter \(t_c\) that has a physical meaning; \(t_c\) and \(\mathcal{H}_{\text{ext}}(t)\) are simultaneously varied by lengthening the
cycle time.

Pulse sequences are usually designed so that \(\mathcal{H}(0)\) has
some particular desired property, and then higher-order
terms are minimized. For example, line narrowing
sequences for solids completely suppress the dipolar or
quadrupolar interaction, but not the chemical shift in-
teraction, in this lowest-order term.\(^11\) If a pulse se-
quence is symmetric, such that \(\mathcal{H}_{\text{int}}(t) = \mathcal{H}_{\text{int}}(t_c - t)\), \(\mathcal{H}(1)\) and all other odd-order correction terms vanish. Sym-
metric pulse sequences are easy to design, so the major contributions to residual linewidths for simple line narrow-
ing sequences come from \(\mathcal{H}(0)\) and from pulse se-
quence imperfections (inhomogeneity, timing errors, and the like). A very powerful method of eliminating these
terms involves alternating between two or more different
cycles (called subcycles) to form a new, larger cycle
which has smaller higher-order terms. Under certain
conditions, some of the higher-order terms for the en-
tire cycle are simply equal to the sum of the corresponding
terms for the subcycles; such terms are said to de-
couple.\(^12\) Decoupled pulse cycles for line narrowing
have been produced that have \(\mathcal{H}(0) = 0\) for the dipolar
Hamiltonian, and have small error terms.\(^12\)

Higher-order terms are usually difficult to calculate,
but for line narrowing sequences their size (and therefore
their contribution to residual line widths) can be esti-
mated. If \(\mathcal{H}(0) = \mathcal{H}(1) = \ldots = \mathcal{H}(n) = 0\), then \(\mathcal{H}(k) = \mathcal{H}^{(k)}\),
where \(\mathcal{H}^{(k)}\) is defined as
\[
\mathcal{H}^{(k)} = \left( \frac{-i}{t_c} \right)^k \int_0^{t_c} dt_{k+1} \int_0^{t_{k+1}} dt_k \ldots \int_0^{t_2} dt_1 \mathcal{H}_{\text{int}}(t_{k+1}) \times \mathcal{H}_{\text{int}}(t_k) \ldots ,
\]
where \(k = n, n + 1, \ldots , 2n\).

Reference \(^9(a)\) contains a weaker version of this theo-
rem, which requires \(\mathcal{H}(j) = 0\) for all \(j < n\) for \(\mathcal{H}(n) = \mathcal{H}^{(n)}\),
but inspection of their proof\(^13\) leads to the immediate
conclusion that \(\mathcal{H}(j) = 0\) for all \(j < (n - 1)/2\) is sufficient
for \(\mathcal{H}(n) = \mathcal{H}^{(n)}\).

The volume of integration is \((t_c)^{n+1}\), so \(\mathcal{H}(n)\) can be
easily estimated in terms of \(\mathcal{H}_{\text{int}}\). To do this estima-
tion we need to use the concept of the norm of a matrix.

We will define the norm of an arbitrary \(N_T \times N_T\) matrix
\(A\) as
\[
||A|| = \left[ \frac{1}{N_T} \text{Tr}(A^*A) \right]^{1/2} .
\]

\(||A||\) is invariant under unitary transformations, so if
\(A\) is Hermitian, \(||A||\) is the root-mean square eigen-
value of \(A\), called \(M_{\text{rms}} (A)\). Other convenient prop-
ties that are easily proven are as follows:

(1) If \(A\) and \(B\) are Hermitian, \(||AB|| = ||BA|| \leq N_T \times N_T ||A|| ||B||\).

(2) \(||I|| = 1\), where \(I\) is the identity matrix.

(3) If \(A\) is Hermitian, \(||A^n||\) is the square root of the
\((2n)\text{th moment of the distribution of the eigenvalues,}
\text{called } M_{\text{rms}} ^{(n)} (A)\). Since \(M_{\text{rms}} ^{(2n)} (A) = [M_{\text{rms}} ^{(n)} (A)]^n\) for any distri-
bution, \(||A^n|| \geq ||A||^n\).

(4) If \(A\) and \(B\) are similar Hermitian matrices, such that
\(A = UBU^\dagger\), \(||AB|| \leq ||A|| ||B||\).

(5) However, if \(A\) and \(B\) are two different matrices,
with nothing else known about either matrix, then
\(||AB|| = \frac{\sum \lambda _{\text{max}} B \lambda _{\text{max}} A}{\lambda _{\text{max}} B \lambda _{\text{max}} A}\) is the sum of \(N_T\) numbers, which we
expect will add randomly. Then \(\langle ||AB||^2 \rangle ^{1/2} \sim N_T \langle (\lambda _{\text{max}} B \lambda _{\text{max}} A) \rangle ^{1/2} \) and this im-
plies \(||AB|| \equiv ||A|| ||B||\).

Properties (2) and (4) imply that \(||\mathcal{H}_{\text{int}}(t_n)|| \times \mathcal{H}_{\text{int}}(t_{n-1}) \ldots \mathcal{H}_{\text{int}}(t_1) \leq \frac{||\mathcal{H}_{\text{int}}(t_1)||^{2n}}{2^n n!} .\) For many sys-
tems the eigenvalues of \(\mathcal{H}_{\text{int}}\) have roughly a Gaussian
distribution, and in this case

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Hamiltonian expansion of the propagator are nk-quantum selective if and only if it is invariant to a \( k \)-fold tensor product of the propagators for the two subcycles, we have

\[
\langle \rho(t) | e^{-i H' t} | \rho(0) \rangle = \sum_{j=1}^{n} \langle \rho(0) | J_{1j}^{(1)} J_{2j}^{(2)} \cdots J_{nj}^{(n)} | \rho(0) \rangle 
\]

where the terms represented by (\ldots) are products of three or more operators, multiplied by \( \langle \rho(0) \rangle^3 \) or higher powers of \( \langle \rho(0) \rangle \). \( \rho^{(n)}(t) \) can only appear in the first term on the l.h.s., and all other terms must have smaller super-scripts. By assumption, \( \rho^{(n)}(t) \) is nk-quantum selective for all \( k \leq j \). When \( k = 0 \), Eq. (11) simplifies to

\[
E^{(0)}(t) = (E^{(0)} e^{i J_{1j}^{(1)} J_{2j}^{(2)} \cdots J_{nj}^{(n)}})
\]

so \( E^{(0)}(t) \) is nk-quantum selective if \( j \geq 0 \). It follows by induction that all the operators \( \rho^{(n)}(t) \) are nk-quantum selective by considering progressively higher powers of \( t \), through \( (t_j)^{j+1} \).

The only possible ns term proportional to \( (t_j)^{m+1} \) on the l.h.s. is then \( -E^{(1)}(t) t_{m+1} \), since all other terms involve only lower-order operators which are nk-quantum selective. Similarly, the only possible ns term on the
The only property of \( nk \)-quantum selective operators that was needed to prove Theorem I was closure of this set of operators under addition and multiplication, and similar theorems can be proven for any other set of operators with these two closure properties. In particular, the null set is closed under addition and multiplication. A decoupling theorem for this case (i.e., \( \mathcal{K}^{(0)} = \mathcal{K}^{(1)} = \cdots = \mathcal{K}^{(N = 0)} \)) was proven by Burum and Rhim. Another set which satisfies the required closure properties is the set of \( N \)-quantum operators in an \( N \) spin-\( \frac{1}{2} \) NMR system, plus the populations of the two extreme sates; this case will be discussed later.

The size of the first \( nns \) term for a \( j \)-order \( nk \)-quantum selective sequence can be readily estimated.

**Theorem II:** If a sequence is \( j \)-order \( nk \)-quantum selective, the non-\( nk \)-quantum selective (\( nns \)) part of \( \mathcal{K}^{(j \mu)} \) can be written as

\[
\mathcal{K}^{(j \mu)} = \left\{ \left( -1 \right)^{i \mu} \frac{e \mu}{\mu} \right\} \times \left[ \int_{t_0}^{t_2} dt_{\mu} \cdots \int_{t_0}^{t_{j \mu}} dt_{\mu_2} \cdots \int_{t_0}^{t_2} \mathcal{K}_{\mu_1} \mathcal{K}_{\mu_2} \cdots \mathcal{K}_{\mu_j} \right]_{\text{sym}} \tag{16}
\]

**Proof:** The proof of this theorem is identical to the proof that \( \mathcal{K}^{(j \mu)} \) has this form if \( \mathcal{K}^{(0)} = 0 \) for all \( i \neq j \), contained in Ref. 9(a), so we will merely outline it. The most general expression for the term proportional to \( (\mu_{\mu})^{2j+1} \) in the propagator is

\[
\left( -1 \right)^{j \mu} \int_{t_0}^{t_2} dt_{\mu} \cdots \int_{t_0}^{t_{j \mu}} dt_{\mu_2} \cdots \int_{t_0}^{t_2} \mathcal{K}_{\mu_1} \mathcal{K}_{\mu_2} \cdots \mathcal{K}_{\mu_j}
\]

\[
\times dt_{\mu_1} \mathcal{K}_{\mu_1} (t_{\mu_1}) \cdots \mathcal{K}_{\mu_j} (t_{\mu_j}) \tag{17}
\]

Expanding \( U = \exp \left\{ -i \left( \mathcal{K}^{(0)} + \mathcal{K}^{(1)} + \cdots + \mathcal{K}^{(N)} \right) \right\} \) as in the lhs of Eq. (11), the only possible non-\( nk \)-quantum selective term proportional to \( (\mu_{\mu})^{2j+1} \) is \( \left( - \mathcal{K}^{(j \mu)} \right)_{\text{sym}} \), which proves the theorem.

It should be noted that Eq. (16) is only valid for the first \( nns \) term, while if \( \mathcal{K}^{(0)} = 0 \) for all \( n < j \) a similar expression holds for all terms up to \( \mathcal{K}^{(0)} \). The difference is that the lhs of Eq. (11) contains operators such as \( \mathcal{K}^{(k_1)} \mathcal{K}^{(k_2)} \), which vanish if either \( \mathcal{K}^{(k_1)} \) or \( \mathcal{K}^{(k_2)} \) vanishes, but which are generally \( nk \)-quantum selective only if both \( \mathcal{K}^{(k_1)} \) and \( \mathcal{K}^{(k_2)} \) are \( nk \)-quantum selective.

**C. Design of selective sequences**

**1. Zero-order selective sequences**

Starting from any cyclic sequence of pulses and delays, we can produce a sequence which is zero-order \( nk \)-quantum selective, using a technique which we call phase cycling. This is illustrated in Fig. 4(a). We assume that the cyclic sequence has a duration \( \Delta T \) (which we call a subcycle), an effective Hamiltonian \( \mathcal{K}_{\mu} = \mathcal{K}^{(0)} + \mathcal{K}^{(1)} + \cdots + \mathcal{K}^{(N)} \) and a propagator \( U_{\mu} = \exp(-i\mathcal{K}_{\mu} \Delta T) \). At the end of the interval \( \Delta T \), the sequence is repeated with all radiation phase shifted by \( \phi = 2\pi/n \) about the \( z \) axis, giving a new effective Hamiltonian \( \mathcal{K}_{\phi} \) and propagator \( U_{\phi} \). \( \mathcal{K}_{\phi} \) is related to \( \mathcal{K}_{\mu} \) by a rotation of \(-\phi\) about the \( z \) axis:

\[
\mathcal{K}_{\phi} = \exp(i\phi L_z) \mathcal{K}_{\mu} \exp(-i\phi L_z) \tag{18}
\]

\[
(\mu_{\phi, \mu})_{ij} = (\mu_{\mu, \mu})_{ij} \exp(i\phi (m_i - m_j)) \tag{19}
\]

and \( U_{\phi} \) is related to \( U_{\mu} \) exactly in the same manner.

This phase shift is repeated \( n \) times, creating a cycle with cycle time \( t_c = n \Delta T \). We would like to calculate \( \mathcal{K}^{(1)} \) for this cycle. Since \( t_c \) is proportional to \( \Delta T \), it is clear that \( \mathcal{K}^{(0)} \) and \( \mathcal{K}^{(1)} \) scale in exactly the same manner when \( t_c \) is changed, and by equating terms proportional to \( t_c \) with those proportional to \( \Delta T \), we find

\[
\mathcal{K}^{(1)} = \frac{1}{n} \sum_{m=1}^{n} \mathcal{K}^{(0)} = \frac{1}{n} \sum_{m=1}^{n} \exp(itcL_z) \mathcal{K}^{(0)} \exp(-itcL_z) \tag{20}
\]

This sum scales the matrix element \( \langle \mu_{\phi, \mu} \rangle_{ij} \) by \( 1/n \) \( \times \sum_{m=1}^{n} \exp(i2\pi m/j) \), where \( p = m_i - m_j \); this scaling factor is zero unless \( p = nk \). Therefore, \( \mathcal{K}^{(0)} \) is a pure \( nk \)-quantum selective operator. Since \( \mathcal{K}^{(0)} \) decouples, any other permutation of the subcycles is also acceptable. Higher order terms have some \( nk \)-quantum selective parts (for example, there is a contribution \( \sum_{m=1}^{n} \mathcal{K}^{(0)} \) to \( \mathcal{K}^{(1)} \)) but no other higher order terms are completely selective. Thus, the sequence obtained by phase cycling is zero-order \( nk \)-quantum selective.

If \( t_c \) can be made arbitrarily small, all the higher-order terms in the average Hamiltonian expansion become unimportant, and a zero-order selective sequence becomes completely selective; of course, the selective term \( \mathcal{K}^{(0)} t_c \) is not zero, but this can be remedied by repeating the zero-order sequences many times. In general, however, \( t_c \) cannot be made arbitrarily small, so higher-order selectivity is desirable. One simple way to get a first-order selective sequence is to symmetrize the cycle, as illustrated in Fig. 4(b). \( \mathcal{K}^{(1)} \) is still \( nk \)-quantum selective, and the symmetrization causes \( \mathcal{K}^{(1)} \) to vanish for all odd \( j \), so the first non-selective term is \( \mathcal{K}^{(2)} \).

**2. Sequences selective to arbitrary order**

Suppose that the sequence for \( \mathcal{K}_{\mu} \) in Fig. 4(a) is already \( j \)-order \( nk \)-quantum selective, instead of being
nonselective as was assumed earlier. Theorem I proves that the sequence obtained by phase cycling is \((j + 1)\)-order \(nk\)-quantum selective, because \((\mathcal{R}^{(nk)})_\text{ase}\) de-couples:

\[
(\mathcal{R}^{(nk)})_{\text{ase}} = \sum_{j=0}^{n-1} (\mathcal{R}^{(j+1)})_{\text{ase}}
\]

\[
= \sum_{j=0}^{n-1} [\exp(\imath \phi J_\phi) \mathcal{R}^{(j+1)} \exp(-\imath \phi J_\phi)]_{\text{ase}} = 0. \quad (21)
\]

Therefore, starting from a nonselective \(\mathcal{X}_0\), \((j + 1)\) phase cyclings produce a sequence that is \(j\)-order \(nk\)-quantum selective, requiring \(n^{(j+1)}\) subcycles; each block of \(n\) subcycles is zero-order \(nk\)-quantum selective, each block of \(n^2\) subcycles is \(j\)-order \(nk\)-quantum selective, and so forth. For example, a first-order \(4k\)-quantum selective sequence may be constructed from \(4^2 = 16\) subcycles, and the phases of the subcycles can be written schematically (0: \(\phi = 0\), 1: \(\phi = \pi/2\), 2: \(\phi = \pi\), 3: \(\phi = 3\pi/2\)) as \((0123)(1230)(2301)(3012)\) as \((0123)(1230)(2301)(3012)\) (0123)(1230)(2301)(3012). Each group in parentheses is a zero-order \(nk\)-quantum selective sequence, and is phase shifted by \(\pi/2\) to produce the next group.

In the absence of relaxation, there is no limit to the number of times phase cycling can be applied, and there are sequences which are selective to arbitrarily high order can be designed. In any real system, only a limited number of subcycles could be completed before relaxation effects make the average Hamiltonian calculation invalid. One way to reduce the number of subcycles required to achieve a given order of selectivity is to combine phase cycling and symmetrization, as in Fig. 4(b). The sequence is first-order \(nk\)-quantum selective even if \(\mathcal{X}_0\) is nonselective. If \(\mathcal{X}_0\) is already \(j\)-order \(nk\)-quantum selective \((j\) odd), the phase cycling and symmetrization requires \(2n\) subcycles to make a \((j + 2)\)-order \(nk\)-quantum selective sequence, instead of the \(n^2\) subcycles required for two phase cyclings. Thus, a \((2j + 1)\)-order \(nk\)-quantum selective sequence requires \((2n)^j\) subcycles \((j + 1)\) phase cyclings and \((j + 1)\) symmetrizations] and a \((2j)\)-order \(nk\)-quantum selective sequence requires \((2n)^j\) subcycles \((j + 1)\) phase cyclings and \(j\) symmetrizations. For example, a third-order \(4k\)-quantum selective sequence requires \((2n)^3 = 64\) subcycles, and the relative phases can be written schematically as \((0123)(3210)(0321)(2103)(2103)(3012)(0321)(0123)(3210)\).

III. APPLICATION OF SELECTIVE SEQUENCES TO MULTIPLE-QUANTUM NMR

A. Motivation

In this section the general principles of selective excitation are elaborated for the particular case of a system of directly dipole coupled nuclear spins. The analysis of the spectral line positions of such a system supplies structural information on the lengths and orientations of internuclear vectors and analysis of the relaxation times provides information on molecular motion. Neither analysis can be completed unless the spectrum contains resolved assignable lines. The high resolution and simplicity of the high multiple-quantum spectra make them preferable or indispensable for these analyses. It will be shown how the use of selective sequences promises to overcome the difficulty of small signal intensity, thus making these analyses possible in large spin systems.

The Hamiltonian for the \(N\) spins \(1/2\) of an oriented molecule in a large magnetic field can be written in the rotating frame (in units of \(\hbar = 1\)) as

\[
\mathcal{X}_\text{sat} = \mathcal{X}_{\text{aw}} + \mathcal{X}_{\text{off}} + \mathcal{X}_\text{eff}. \quad (22)
\]

\[
\mathcal{X}_{\text{aw}} = \Delta \omega I_\phi \text{ is the resonance offset term, } \mathcal{X}_{\text{off}} = \sum_{\phi} \mathcal{O}_\phi I_\phi \text{ is the secular portion of the dipolar Hamiltonian, and } \mathcal{X}_\text{eff} \text{ is the time-dependent interaction between the molecules and applied radiation. Other interactions such as chemical shifts or scalar spin-spin couplings may also be accounted for, but we will not consider them here.}

The task we set out to accomplish in this section is to construct a rapidly convergent effective Hamiltonian for an excitation cycle which will contain, in its leading terms, operators \(T_\phi\) with rank \(j\) up to the maximum value \(j = N\) and components \(nk \leq j\). Such a Hamiltonian will create high quantum coherences selectively even if it acts only for short times such that \(\mathcal{X}_\text{eff} \ll 1\). The theoretical tool to be used is a nested series of average Hamiltonian expansions with each successive expansion involving a longer cycle time. This approach is exact when the effective Hamiltonian of each of the preceding expansions is used as the starting point for the next expansion. In practice, one designs sequences with rapidly convergent expansions and proceeds by using only the first few terms as an approximation to the effective Hamiltonian. Precedents for this procedure exist. It is implicitly used whenever a rotating frame Hamiltonian is used as a starting point for an average Hamiltonian expansion. Another example is the "second averaging" procedure used to explain off-resonance effects in multiple pulse line narrowing experiments.

The expansions which were the subject of the previous section were the final expansions, in which the effective Hamiltonians \(\mathcal{X}_k\) for the subcycles were assumed to be known. Figure 5 shows schematically how the phase

\[
\begin{align*}
&|X_0\rangle \rightarrow |X_1\rangle \rightarrow |X_2\rangle \rightarrow |X_3\rangle \\
&|X_1\rangle \rightarrow |X_2\rangle \rightarrow |X_3\rangle \\
&|X_2\rangle \rightarrow |X_3\rangle \rightarrow |X_4\rangle
\end{align*}
\]

FIG. 5. The analogy between a phase cycled sequence consisting of subcycles with effective Hamiltonians \(\mathcal{X}_k\) and a pulse sequence on a hypothetical system with \(\mathcal{X}_\text{stat} = \mathcal{X}_0\). Average Hamiltonian theory can be applied to the pulse sequence.

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FIG. 6. Possible pulse sequences for the subcycles in Fig. 4. In Fig. 6(a), time reversal sequences generate \( \mathcal{H}_T^* T = -\mathcal{H}_T \), so that \( \| \mathcal{X}_e \| \ll \| \mathcal{X}_{e_{\text{sub}}} \| \), but \( \mathcal{X}_e \) contains multiple-quantum coherences. In Fig. 6(b), a WLUHA sequence with a long cycle time has the same effect.

shifts of the cycle may be viewed as a series of \( z \) pulses on a system with an otherwise time independent Hamiltonian \( \mathcal{H}_{e_{\text{sub}}} \). In the next section the subcycles are treated.

B. The design of effective subcycles

If a subcycle can be designed which has high quantum operators in the leading terms of its effective Hamiltonian \( \mathcal{H}_{e_{\text{sub}}} \), then the cycle need only be selective to a low order since high quantum operators will appear in its leading terms. Since \( \mathcal{X}_{e_{\text{sub}}} \) contains only first and second rank operators, no simple rotation of \( \mathcal{X}_{e_{\text{sub}}} \) can contain high quantum operators. It is only through the action of the couplings between spins that such operators can appear in the evolution, so it appears desirable that the subcycle last at least long enough that \( \| \mathcal{X}_{e_{\text{sub}}} \Delta T \| \sim 1 \). An apparent dilemma arises here though, for if the cycle consists of many subcycles, then \( \| \mathcal{X}_{e_{\text{sub}}} \Delta T \| \geq 1 \) and convergence of the effective Hamiltonian expansion becomes doubtful. The solution to this problem is to arrange the internal structure of the subcycles so that \( \| \mathcal{X}_{e_{\text{sub}}} \Delta T \| \ll 1 \) but \( \| \mathcal{X}_e \Delta T \| \ll 1 \). Several approaches will be discussed. One general approach to subcycle design uses the method of time reversal and is illustrated in Fig. 6(a). \( \mathcal{X}_e \) and \( \mathcal{X}_{e_{\text{sub}}} \) are approximations to the effective Hamiltonians for the periods \( T \) and \( T' \), respectively. Before examining particular sequences for the periods \( T \), \( T' \), and \( \Delta T' \), we indicate how time reversal leads to the desired conditions that \( \mathcal{X}_e \) contains high quantum operators and that \( \| \mathcal{X}_e \Delta T \| \ll 1 \).

Pulse sequences can be designed with \( \mathcal{X}_0^{(6)} = \frac{1}{2} (\mathcal{X}_{p_{xx}} + \mathcal{X}_{p_{yy}}) = -\frac{1}{2} \mathcal{X}_{p_{xx}} \), and the effect of such sequences is to make the spin system appear to evolve backwards in time. If such a pulse sequence is applied for a time \( 2T \) and then turned off, the internal condition will return after a time \( T \). If \( \| \mathcal{X}_{e_{\text{sub}}} \| T \geq 1 \), both the forward time and reversed time propagators can contain irreducible tensor operators of arbitrarily high rank, but will commute with \( I_e \). Similarly, pulse sequences can be designed with \( \mathcal{X}_0^{(6)} = \frac{1}{2} (\mathcal{X}_{p_{xx}} + \mathcal{X}_{p_{yy}}) = -\frac{1}{2} \mathcal{X}_{p_{xx}} \). In this case, the propagator obtained from a sequence with \( \| \mathcal{X}_{e_{\text{sub}}} \| T \geq 1 \) will contain irreducible tensor operators of arbitrarily high rank, but will not commute with \( I_e \). Such a propagator can generate multiple-quantum coherences and can be viewed as a multiple-quantum rotation (as opposed to the rotation produced by a single strong pulse, which will only generate single-quantum coherences from a density matrix proportional to \( I_e \)).

The propagator for the subcycle of Fig. 6(a) is \( \mathcal{X} = \exp(-i \mathcal{X}_e T') \exp(-i \mathcal{X}_{e_{\text{sub}}} T) \). Time reversal techniques may be used to arrange that \( \mathcal{X}_e^* T' = -\mathcal{X}_e T \). When this condition holds, the periods \( T \) and \( T' \) may be viewed as a complementary pair of multiple-quantum rotations which sandwich the period \( \Delta T' \). Together they form a cycle and the average Hamiltonian for this cycle is

\[
\mathcal{X}_0^{(0)} = \frac{1}{\Delta T'} \int_0^{\Delta T'} \mathcal{X}_0^{(0)}(t) \, dt = \frac{\Delta T'}{\Delta T} e^{i\mathcal{X}_0 T} \mathcal{X}_0 e^{-i\mathcal{X}_0 T}.
\]  

If \( \mathcal{X}_e \) is nonscalar, bilinear, and does not commute with \( \mathcal{X}_{e_{\text{sub}}} \), then \( \mathcal{X}_0^{(0)} \) will contain high multiple-quantum operators when \( \| \mathcal{X}_e \| T \geq 1 \). Since the exponential operators constitute a unitary transformation, the norm of \( \mathcal{X}_e \) is conserved and \( \| \mathcal{X}_0^{(0)} \| = (\mathcal{X}_e T / \mathcal{X}_{e_{\text{sub}}} T) \| \mathcal{X}_e \| \). The desired effect of reducing the norm of the subcycle Hamiltonian is achieved when \( \mathcal{X}_e T \ll \mathcal{X}_{e_{\text{sub}}} T \). Since \( \| \mathcal{X}_0^{(0)} \| = \| \mathcal{X}_e \| \| \Delta T' \| \), the small interval \( \Delta T' \) may be thought of as an effective cycle time for the subcycle. This concept is useful in that lengthening the cycle time corresponds to lengthening \( \mathcal{X}_e \) and not \( T \) or \( T' \).

Several choices are possible for \( \mathcal{X}_e \), \( \mathcal{X}_{e_{\text{sub}}} \), and \( \mathcal{X}_{e_{\text{sub}}} \). Let \( \mathcal{X}_e = \mathcal{X}_{p_{xx}} = \mathcal{X}_{p_{yy}} = \mathcal{X}_{p_{zz}} = \mathcal{X}_p(\mathcal{X}_{p_{xx}} \mathcal{X}_{p_{zz}} - 1, I_1) \) produced by the sequence \( 90^\circ_x - T - 90^\circ_y - T - 90^\circ_y - T - 90^\circ_x - T - 90^\circ_z - T \), repeated enough times to fill a period \( T' = 2T \); and \( \mathcal{X}_{e_{\text{sub}}} = \mathcal{X}_{p_{xx}} \) using no pulses at all (\( \mathcal{X}_e \) is a "window" in the sequence). The particular time reversing sequence chosen for \( \mathcal{X}_e \) has \( \mathcal{X}_0^{(0)} = \frac{1}{2} \mathcal{X}_{p_{xx}} \); it is symmetric, so \( \mathcal{X}_0^{(1)} = 0 \). Using the notation of Ref. 11 for various error terms from pulse imperfections, we find \( \mathcal{X}_0^{(6)} \) (resonance offset and chemical shift terms) = 0; \( \mathcal{X}_0^{(9)} \) (rf inhomogeneity effects) = 0 (to order \( \epsilon \)); \( \mathcal{X}_0^{(6)} \) (nonzero pulse width) = 0; and \( \mathcal{X}_0^{(1)} = \mathcal{X}_0^{(1)} = \mathcal{X}_0^{(1)} = 0 \) can be achieved by symmetrization. Neglecting all correction terms, we have

\[
\mathcal{X}_0^{(0)} \Delta T' = \exp(i\mathcal{X}_e T)(\mathcal{X}_{e_{\text{sub}}} \Delta T') \exp(-i\mathcal{X}_e T),
\]

which conserves the even or odd quantum character of the initial density matrix. This is sometimes convenient; for example, a third order 10k-quantum selective sequence requires \( \mathcal{X}^{(2)} = 400 \) subcycles, but a third order 5k-quantum selective sequence only requires 100 subcycles, and if no odd-quantum coherences are present in \( \mathcal{X}_e \), the two sequences have the same effect.

The sequence \( 90^\circ_x - T - 90^\circ_y \) only gives \( \mathcal{X}_e = \mathcal{X}_{p_{xx}} \) if there are no chemical shifts, if \( \Delta \omega = 0 \), if the rf homogeneity and the static homogeneity are perfect, and if the pulse widths are negligible. Thus, even if \( \tau \) is kept short in the sequences for \( T' \), it may be that \( \| \mathcal{X}_e \| \Delta T' \| \)
FIG. 7. Pulse sequences for multiple-quantum experiments. (a) is the general case, with preparation propagator $U$ and mixing propagator $V$. (b) is the simplest nonselective multiple-quantum experiment. (c) is formally equivalent to (b) and reveals the symmetry between preparation and mixing. (d) is a fully selective experiment, involving selective preparation, mixing, and detection.

> \| \mathcal{X}_w \Delta \tau \| \text{ because neglected error terms would enter. One very convenient way to lessen the severity of error terms is to design a sequence with an effective Hamiltonian having only double-quantum terms; then time reversal can be achieved by a phase shift. For example, the sequence } \\
> \left( \frac{1}{2} \begin{pmatrix} \tau - 90 & \tau - 90 & \tau - 90 & \tau - 90 \\ \tau - 90 & \tau - 90 & \tau - 90 & \tau - 90 \\ \tau & \tau & \tau & \tau \\ \tau & \tau & \tau & \tau \end{pmatrix} \right) \\
> \text{ has an average Hamiltonian } \mathcal{X}_D^\text{avg} = (\tau' + \tau + \tau') \mathcal{X}_{D,zz} + (\tau' + \tau + \tau') \mathcal{X}_{D,xx} \text{ in the limit of } 0 \text{-function pulses, and if } \tau' = 2 \tau, \text{ then} \\
> \mathcal{X}_D^\text{avg} = \frac{1}{2} (2 \mathcal{X}_{D,zz} + \mathcal{X}_{D,xx}) = \frac{1}{2} (\mathcal{X}_{D,zz} - \mathcal{X}_{D,xx}). \quad (25) \\
> \text{ This is a pure double-quantum operator, which can be inverted by phase shifting the sequence by } 90^\circ. \text{ If the pulses are assumed to have a square envelope but a finite width } \tau', \text{ } \mathcal{X}_D^{\text{avg}} \text{ is a pure double-quantum operator for } \tau' = 2 \tau + \tau'; \text{ other pulse errors may change this relation slightly.} \\
> \text{ The sequence can be used with } x \text{ and } \bar{z} \text{ pulses with enough cycles to create a total length } T \text{ such that } \| \mathcal{X}_w \| T \geq 1. \text{ } \mathcal{X}_w \text{ is provided by the same sequence, except with } y \text{ and } \bar{y} \text{ pulses. Potential sequences for } \mathcal{X}_w \text{ include the following:} \\
> \text{ (1) no pulses, giving } \mathcal{X}_w = \mathcal{X}_{D,zz} \text{ and even-quantum selection for initial condition } \beta_I; \\
> \text{ (2) } 45_\theta - \Delta \tau'_z - 45_\theta, \text{ giving all orders in } \mathcal{X}_w; \\
> \text{ (3) the same sequence as } \mathcal{X}_p, \text{ except phase shifted by } 45^\circ; \text{ clearly } \mathcal{X}_w \text{ is also a pure two-quantum operator, but } \{ \mathcal{X}_p, \mathcal{X}_w \} = 0, \text{ so multiple-quantum coherences still develop; after a brief interval } \Delta \tau'_z, \text{ another phase shift of } 45^\circ \text{ gives } \mathcal{X}_w'. \text{ Experimental work to date has used } \mathcal{X}_w = \mathcal{X}_{D,zz} \text{ and } \mathcal{X}_w' = \frac{1}{2} (\mathcal{X}_{D,yy} - \mathcal{X}_{D,xx}). \text{ Figure 3 is the } 4k \text{-quantum selective spectrum of oriented benzene obtained with this approach. Previous attempts with } \mathcal{X}_w = \mathcal{X}_{D,zz} \text{ and } \mathcal{X}_w' = -\frac{1}{2} \mathcal{X}_{D,xx}, \text{ as discussed above, were less successful.} \\
> \text{ The standard WAHUHA sequence, illustrated in Fig. 6(b), is another possible pulse sequence for } \mathcal{X}_w. \text{ If } \mathcal{X}_\text{int} = \mathcal{X}_{D,zz}, \text{ this sequence gives } \mathcal{X}_\text{int} = 0 \text{ (neglecting pulse errors). However, if } \| \mathcal{X}_{D,zz} \| T \geq 1, \text{ } \mathcal{X}_w \text{ will have strong contributions from } \mathcal{X}_\text{int}, \text{ and higher-order terms which contain multiple-quantum coherences. For some value of } T \text{ such that } \| \mathcal{X}_{D,zz} \| T \approx 1, \text{ we expect to find } \| \mathcal{X}_w \| \ll \| \mathcal{X}_{D,zz} \|, \text{ but } \mathcal{X}_w \text{ will contain a substantial fraction of multiple-quantum coherences. The sequence is repeated } N \text{ times, so } \Delta \tau'_z = 6NT. \text{ When such a subcycle is incorporated into a selective excitation sequence, it will prove useful to think of } T \text{ as a fixed parameter while } N \text{ is varied in order to vary the cycle time.} \\
> \text{ Clearly, any other line narrowing sequence is also a candidate for producing } \mathcal{X}_w, \text{ but this sequence would probably be the easiest to use because of its relatively large correction terms. A possible advantage over the use of time-reversing sequences is the very low duty cycle, which results because } T \text{ is much longer than in a normal WAHUHA experiment.} \\
> \text{ C. Selective sequences in the multiple-quantum NMR experiment} \\
> \text{ Any selective sequence can be incorporated into the general framework of a multiple-quantum experiment, shown in Fig. 7(a). A large static magnetic field } B = B_0 \bar{z} \text{ is applied to a system of } N \text{ spin-1 nuclei, giving an equilib- rium spin density matrix } \rho_0 = \exp(-\beta' I_\bar{z})/T \text{ in the high temperature approximation we write the reduced density matrix as } \rho_0 = (2I + 1)^{-N}(-\beta' I_\bar{z}) = \beta_0 I_\bar{z}. \text{ A pulse sequence (which need not be cyclic) is applied to produce multiple-quantum coherences. This is called the preparation sequence, and we will denote its propagator by } U. \text{ The spins then evolve under } \mathcal{X}_\text{int} \text{ for a time } t_1. \text{ In the simplest experiments no pulses are applied during } t_1; \text{ however, decoupling, spin echoes, or more complicated sequences are possible if suppression of part of } \mathcal{X}_\text{int} \text{ is desired. Because only the operators } I_x \text{ and } I_y \text{ are observable, another pulse sequence (called the mixing sequence) is used to transfer the multiple-quantum coherences that evolved during } t_1 \text{ into single-quantum coherences; we will denote the propagator for the mixing sequence by } V. \text{ The oscillating magnetization proportional to } \langle \bar{z}_n \rangle \text{ is measured, the experiment is repeated many times with different values of } t_1, \text{ and the signal is Fourier transformed with respect to } t_1 \text{ to produce the multiple-quantum spectra. In this work it will be assumed that the initial condition is } \beta_0 I_\bar{z} \text{ and that only the operators corresponding to magnetization at the end}
of the mixing period \( V \) are detected. The use of coherently excited initial conditions and the implications of selective excitation for full two dimensional NMR experiments will be discussed elsewhere.

In the simplest experiment, \( U \) and \( V \) are nonselective as shown in Fig. 7(b). The first two pulses, separated by a time \( \tau_p \), constitute the preparation sequence. If we further assume that all pulses are strong pulses, so that the internal spin Hamiltonian \( X_\alpha = \Delta \omega I_\alpha + X_\alpha \) can be neglected during the short intervals that the rf field is present, we can write the reduced density matrix at the end of the second pulse as

\[
\rho = \beta \exp(-iH_s \tau_p/2) \exp(-iH_\alpha \tau_p) \exp(iH_s \tau_p/2) 
\]

\[
\times \exp(iH_s \tau_p) \exp(iH_\alpha \tau_p/2) = \beta \exp(iH_s \tau_p) I_s \exp(iH_\alpha \tau_p) ,
\]

(26)

where

\[
X_\alpha = \Delta \omega I_\alpha + X_\alpha \quad (\alpha = x, y, \text{ or } z) ,
\]

(27)

\[
X_\alpha \alpha = \sum_{\ell < j} |I_\alpha I_\alpha - I_\ell I_j \rangle (\alpha = x, y, \text{ or } z) .
\]

(28)

Because the dipolar Hamiltonian is bilinear, \( \rho \) will have multiple-quantum matrix elements if \( \| X_\alpha X_\alpha \| \geq 1 \), in contrast to excitation by a single pulse which produces only single-quantum coherence. The third pulse and the final delay \( t_s \) constitute the mixing and detection sequence, and \( \langle f_s (t_s) \rangle \) is measured. This pulse sequence, modified to include echoes to remove \( \Delta \omega \) and field inhomogeneity from the evolution time \( t_1 \), and phase shifted to separate the different \( n \) values, \( T_2, T_4 \) is the one used for most multiple-quantum experiments.

The dramatic decrease in intensity as \( n \) increases and the low intensity of individual transitions mentioned in the introduction is now easily derived. To simplify the formalism, we note that there is actually a great deal of similarity between the preparation and mixing portions of the pulse sequence, which is hidden by the experimental need to measure \( \langle I_s \rangle \) or \( \langle f_s \rangle \) even though the initial density matrix is proportional to \( I_s \).

If we imagine instead that we observe \( \langle I_s \rangle \), as in Fig. 7(c), we have to insert a fourth pulse at the end of \( I_s \). The sequences in Figs. 7(b) and 7(c) would always give exactly the same spectra, but in Fig. 7(c) the symmetry between preparation and mixing is apparent; in fact, if \( t_s = \tau_p \), the propagators are identical:

\[
U = V = \exp(-iH_s \tau_p/2) \exp(-iI_\alpha \tau_p) \exp(iH_s \tau_p/2) .
\]

(29)

We can write the observed signal as

\[
\langle I_s (t_s , t_1 , t_2 = \tau_p) \rangle = \text{tr}(\rho f_s )
\]

\[
= \beta \text{tr} \left[ V \exp(-iH_\alpha t_1) U \exp(iH_\alpha t_1) V I_s \right] ,
\]

(30)

\[
= \beta \text{tr} \left[ (U I_s U^\dagger) \exp(iH_\alpha t_1)(V I_s V^\dagger) \exp(-iH_\alpha t_1) \right] ,
\]

(31)

\[
= \beta \sum_{m \neq n} (U I_s U^\dagger)_{mn} (U^\dagger I_s U)_{mn} e^{i\omega_{mn} t_1} ,
\]

(32)

where \( \omega_{mn} \) is the energy difference between the states \( m \) and \( n \). The maximum signal is obtained when \( |(U I_s U^\dagger)_{mn}| = |(U^\dagger I_s U)_{mn}| \) for every matrix element, and it can easily be shown that the condition is satisfied if \( U \) has the form in Eq. (29)\(^3\); in fact, for this sequence \( (U I_s U^\dagger)_{mn} = (U^\dagger I_s U)_{mn} \). Therefore, the intensity of the multiple-quantum transition at the frequency \( \omega_{mn} \) is equal to \( \beta |(U I_s U^\dagger)_{mn}|^2 \) assuming that all peaks are resolvable. The total intensity of the spectrum, which is divided among all the orders of multiple-quantum transitions, is equal to \( \beta \text{tr}[(U I_s U^\dagger)^2] \). However, this quantity is invariant to unitary transformation and is the same as the integrated intensity of the single-quantum spectrum in a conventional one-pulse experiment \( \beta \text{tr}(U^2) \). Therefore, the total spectral intensity of the nonselective experiment is fixed.

If the multiple-quantum matrix elements are inefficiently excited (for example, if \( \| X_\alpha X_\alpha \| < 1 \)), \( UI_s U^\dagger \) has large matrix elements along its diagonal. These matrix elements are populations, so they do not evolve, and most of the intensity of the multiple-quantum spectrum is found at \( \omega = 0 \). An efficient wide band nonselective excitation has little intensity at \( \omega = 0 \), and excites all of the possible transitions about equally. Therefore, the average intensity of a single line in a multiple-quantum spectrum is smaller than the average intensity of a single line in an ordinary single pulse experiment, by a ratio (number of single-quantum transitions)/(number of excited multiple-quantum transitions). When there are many spins, the intensity of a single transition becomes extremely small. For example, a system with \( N \) spins \( \frac{1}{2} \) and no symmetry has \( 2^{2N} \) possible distinct matrix elements, so totally nonselective excitation gives a signal for each transition of \( 2^{2N} |\beta \text{tr}(U^2)| \).

If only certain orders of multiple-quantum transitions are excited, but the excitation is still efficient (in the sense that the peak at \( \omega = 0 \) is small), the intensity of a single transition grows. For example, if the resonance offset is removed from the excitation and detection periods (by echoes, for example), \( H_s \) retains only zero- and two-quantum operators, and only even-quantum coherences are excited. Since roughly half the coherences are even quantum, this process increases the intensity of an even quantum transition by a factor of 2. If only a few transitions are excited (by an extremely selective sequence) and the sequence is efficient, the intensity of each transition could be enormous. Suppose that selective excitation is used for both preparation and mixing as in Fig. 7(d), and that \( UI_s U^\dagger \) and \( VI_s V^\dagger \) could be prepared with all the matrix elements zero except for the single coherence with \( \Delta m = +N \) and the single coherence with \( \Delta m = -N \). In that case, the signal gain relative to the nonselective experiment would be \( 2^{2N+1} \). However, the density matrix that results is not related to the initial condition \( I_s \) by a unitary transformation, and therefore it cannot be produced by any sequence that does not include relaxation.

A more reasonable estimate of the maximum possible gain is obtained by finding the maximum possible value of \( |(UI_s U^\dagger)_{ab}| \), where \( |a\rangle \) is the single state with \( m = +N/2 \) and \( |b\rangle \) is the single state with \( m = -N/2 \). We have
\[(U_1U')_{ab} = \sum_i U_{ai}(U_1')_{it}U_{tb}, \quad (33)\]
\[= \sum_i U_{ai}(U_1')_{it}\delta_{tb}, \quad (34)\]
\[= \sum_i U_{ai}(U_1')_{it}^* \delta_{tb}. \quad (35)\]

The maximum can be readily seen to be \(U_{ab} = 1/\sqrt{2}, U_{aa} = 1/\sqrt{2}, U_{ab} = 0, U_{ba} = 1/\sqrt{2}, \) and \(U_{ba} = 0.\) The phases are not unique. Such a propagator concentrations the matrix elements of \(U\) in the states with the largest values of \(|m|\). It couples states \(|a\rangle\) and \(|b\rangle\) only to each other, effectively creating a two-level system. The two-level system has

\[U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \exp(i\pi/2)I^y_{m}. \quad (36)\]

where \(I^y_{m}\) is a fictitious spin-\(\frac{1}{2}\) operator for multiple-quantum coherence.\(^{10}\)

Thus, the maximum possible signal is obtained by a selective 90° pulse, shown schematically in Fig. 8. The signal from this transition is, from Eq. (32), \(\langle U^2 \rangle_{I_0} = \langle N \gamma \rangle \beta^2 2^{-3N} \text{Tr}(U^2)\).\(^{10}\)

The gain when compared to totally nonselective excitation is then

\[G_N = \beta^2 \text{Tr}(U^2) / \beta^2 2^{-3N} \text{Tr}(U^2) = (N/2)^{2N} / (N^2 + N + 1)^{2N}. \quad (37)\]

To achieve this gain, we need a sequence that couples the state \(m = N/2\) only to the state \(m = -N/2\). The effective Hamiltonian for this sequence should be some linear combination of \(I^y_{m}\) and \(I^y_{-m}\). This sequence would be used to create \(U_1U'\) and \(V_1V'\). If the effective Hamiltonian has this form for \(m = \pm N/2\), it can have any form whatsoever for the other levels, and the signal in the N-quantum transition will be unaffected.

Often the \((N-1)\)-quantum or \((N-2)\)-quantum transitions in an \(N\)-spin system are more interesting than the \(N\)-quantum transition, since the \(N\)-quantum transition contains no dipolar information. If \((N-1)\)-quantum selection on \(I_x\) is used, the number of transitions increases to \(2N\) for a system without symmetry. In addition, while it is possible to envision a \(\pi/2\) pulse on a two-level system completely depleting the population difference, in a multilevel system it is very unlikely that all population differences can be eliminated simultaneously. Thus, we expect to also produce zero-quantum transitions and populations in the \(m = \pm N/2\) and \(m = \pm (N/2 - 1)\) manifolds, effectively increasing the total number of pumped matrix elements to \(2^{N^2 + 2}\). Now, however, the available fraction of \(I^x_{m}\) is larger. The result of all of these effects is

\[G_{N+1} = \beta^2 2^{N^2 + 2} \text{Tr}(U^2) / \beta^2 2^{-3N} \text{Tr}(U^2) = 2^{N^2 + 2N - 3N + 4}/(N^4 + N + 1) = 2^N, \quad (38)\]

Values of \(G_N\) and \(G_{N+1}\) for systems without symmetry are listed in Table I. If symmetry is included, all gains are reduced, because fewer transitions are allowed and therefore the system is effectively a collection of smaller systems. All of our calculations are still valid, except that the number of density matrix elements excited and the available fraction of \(I^x_{m}\) should be recalculated using the known symmetry. In general, \(N\)-quantum and \((N-1)\)-quantum transitions must have \(A_1\) symmetry, since the states with \(m = \pm N/2\) have that symmetry. The relevant energy level diagram is not a binomial distribution but instead is the group of \(A_1\) states. The calculations are straightforward, and benzene has been included in Table I to illustrate symmetry effects.

These gains become extremely large for large \(N\). However, the single \(N\)-quantum transition contains only a tiny fraction of the total intensity in the nonselective experiment, and therefore we should calculate the total signal available in the \(N\)-quantum and \((N-1)\)-quantum transitions with and without selectivity. This calculation is done in Table II. We assume that the total number of protons in the sample is kept constant as we change \(N\), and calculate the signal size as a fraction of the total magnetization of the sample. The signal size still decreases as \(N\) increases, but the decrease is much slower in the selective experiment, and Table II indicates that selective excitation should dramatically increase the number.

\[\text{FIG. 8. Schematic illustration of the effect of } N\text{-quantum selective sequences. In terms of an effective two-level system involving only } m = \pm N/2\text{, a selective } 90^\circ \text{ pulse transfers the entire population difference between those two states into } N\text{-quantum coherence, giving a gain relative to nonselective excitation of } N2^N \text{ (see text).}\]

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of molecules which could be studied by multiple-quantum spectroscopy. However, the results in Tables I and II were calculated by assuming that a perfectly selective experiment is possible. Since any actual experiment will deviate from this ideal, we must now consider sequences which are not selective to infinitely high order, and which can have other errors.

**IV. EXTENT OF SELECTIVITY IN NONIDEAL SELECTIVE SEQUENCES**

A. General systems

As mentioned earlier, for any multiple-pulse sequence one expects \( \| \text{Selective} \| \leq \| \text{Nonselective} \| \). In a selective sequence, \( \mathcal{X}_e \) is formally equivalent to \( \mathcal{X}_{\text{stat}}(t) \), as illustrated in Fig. 5. Therefore, as \( \| \mathcal{X}_{\text{stat}}(t_e) \| \rightarrow 0 \), \( \mathcal{X}_e \) becomes the dominant term of \( \mathcal{X} \). The nonselective terms of \( U = \exp(\mathcal{X}(N_t)) \) can be made arbitrarily small in principle by making \( t_e \) very short, while if \( N_t \) is kept constant the selective contribution from \( \mathcal{R}(0) \) is unaffected. In practice, the attainable selectivity is limited by several factors.

1. For technical reasons, \( t_e \) cannot be made arbitrarily short. For example, if each subcycle requires pulses with specified flip angles, each pulse has a finite width which depends on the strength of the exciting field.

2. The time required to pump multiphoton coherences is generally dependent on the "anharmonicity" of the energy level spacing. The excitation sequence needs to extend for a period comparable to the inverse of the anharmonic frequencies, which in the last section were the dipolar frequencies. This problem was investigated in depth in the last section; one solution is to construct a subcycle with an effective Hamiltonian \( \mathcal{X}_{\text{stat}} \ll \mathcal{X}_{\text{stat}} \), so that \( \| \mathcal{X}_{\text{stat}} \| \) can be small even though \( \| \mathcal{X}_{\text{stat}} \| \) is not. If this is not possible, the general considerations of the preceding section still hold, but to retain selectivity the subcycles would need to be shorter, and cycles selective to higher order would be needed to obtain high quantum operators.

3. Because there is a lower limit to the length of a subcycle, the minimum time needed for a \( j \)-order \( n \)-quantum selective sequence increases rapidly as \( j \) increases. However, relaxation mechanisms make the average Hamiltonian calculation invalid if the total length of a sequence is comparable to \( T_2 \), the coherence dephasing time. Thus, for any system only a finite order of selectivity is possible. Inhomogeneous systems are a special case; excitation designed to compensate for such broadening may allow homogeneous selective excitation.

4. Timing errors, inaccurate phase shifts, or other failures in control over coherence will reduce the selectivity of any sequence.

In this subsection we discuss the limitations for general spectroscopic systems. In order to estimate the importance of the first three problems listed above, we calculate in the Appendix the size of the first non-\( n \)-quantum selective operator from a \( j \)-order \( n \)-quantum selective sequence, which is \( (-\mathcal{R}(j+1)_{sk} \text{stat}) \), and we determine plausible conditions for convergence of the average Hamiltonian expansion. Only the results will be summarized here. We combine phase cycling and symmetrization into one operation, which turns a \( (j-2) \)-order \( n \)-quantum selective subcycle into a \( j \)-order \( n \)-quantum selective cycle requiring \( 2m \) subcycles [Fig. 2(b)], assuming perfect phase shifts and no timing errors. The norm of the first \( n \)-ns term for the cycle, which is \( \| \mathcal{R}(n)_{sk} \text{stat} \| \), is shown in the Appendix to be related to that of the first \( n \)-ns term of the \( j \)-th subcycle, which is \( \| \mathcal{R}(j+1)_{sk} \text{stat} \| \):

\[
\| \mathcal{R}(j+1)_{sk} \text{stat} \| \approx \| \mathcal{R}(n)_{sk} \text{stat} \| \| \mathcal{R}(n)_{sk} \text{stat} \| \]

\[
\| \mathcal{R}(j+1)_{sk} \text{stat} \| \approx \| \mathcal{R}(n)_{sk} \text{stat} \| \| \mathcal{R}(n)_{sk} \text{stat} \|\]

**TABLE II. Intensity of high multiple-quantum transitions, with and without selectivity, relative to total magnetization of the sample.**

<table>
<thead>
<tr>
<th>( N )</th>
<th>Symmetry</th>
<th>Nonselective</th>
<th>( xG_{n} )</th>
<th>( xG_{2n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>Benzene ( A_1 )</td>
<td>0.187</td>
<td>9.38</td>
<td>3.09</td>
</tr>
<tr>
<td>6</td>
<td>None</td>
<td>0.024</td>
<td>9.38</td>
<td>0.80</td>
</tr>
<tr>
<td>8</td>
<td>None</td>
<td>1.5x10(^3)</td>
<td>5.13</td>
<td>0.24</td>
</tr>
<tr>
<td>10</td>
<td>None</td>
<td>9.5x10(^5)</td>
<td>0.96</td>
<td>0.066</td>
</tr>
<tr>
<td>12</td>
<td>None</td>
<td>5.9x10(^6)</td>
<td>0.29</td>
<td>0.017</td>
</tr>
<tr>
<td>14</td>
<td>None</td>
<td>3.7x10(^7)</td>
<td>0.085</td>
<td>4.5x10(^3)</td>
</tr>
<tr>
<td>16</td>
<td>None</td>
<td>2.3x10(^8)</td>
<td>0.024</td>
<td>1.2x10(^3)</td>
</tr>
<tr>
<td>18</td>
<td>None</td>
<td>1.4x10(^9)</td>
<td>0.0069</td>
<td>3.0x10(^4)</td>
</tr>
</tbody>
</table>

**TABLE III. Values of \( F(n) = [(8/15)n^2 - (2/3)n^3 - (2/15)n^4]^{1/2}/8n^2 \) and \( F(n)_{sk} \text{stat} \).**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( F(n) )</th>
<th>( F(n)_{sk} \text{stat} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.090</td>
<td>11.01</td>
</tr>
<tr>
<td>4</td>
<td>0.044</td>
<td>23.00</td>
</tr>
<tr>
<td>5</td>
<td>0.036</td>
<td>27.00</td>
</tr>
<tr>
<td>6</td>
<td>0.032</td>
<td>31.00</td>
</tr>
<tr>
<td>10</td>
<td>0.029</td>
<td>35.00</td>
</tr>
<tr>
<td>12</td>
<td>0.026</td>
<td>38.00</td>
</tr>
<tr>
<td>14</td>
<td>0.024</td>
<td>41.00</td>
</tr>
<tr>
<td>16</td>
<td>0.023</td>
<td>44.00</td>
</tr>
<tr>
<td>18</td>
<td>0.022</td>
<td>46.00</td>
</tr>
</tbody>
</table>

J. Chem. Phys., Vol. 73, No. 5, 1 September 1980
trix element of $\mathcal{S}$ and a typical matrix element of $(\mathcal{S}^{(i+1)})_{\text{max}}$. At the limit of convergence $\|\mathcal{S}^{(i+1)}t_e - F(n)\|^2$ we find

$$S = K^{-1} F(n)^{1/2} (2n)^{1/2} e^{\gamma/\phi} \left\{ \frac{n^\alpha K}{8K} \right\}^{1/2} \left\{ \frac{n^\alpha K}{12} \right\}^{1/2} \right\}^{1/2} \right\}^{1/2},$$

(41)

where $K$ is defined as the total number of allowed transitions divided by the total number of $nk$-quantum transitions, and $\alpha$ is defined by the relation $\|\mathcal{S}\|^2 = K \|\mathcal{S}\|^2$ [see Eq. (20)]; the reason for the definition is that if all the matrix elements of $\mathcal{S}$ have roughly equal magnitude, $\alpha = 1$.

To go further, we need to know the relative sizes of $\|\mathcal{S}\|^2$, $\|\mathcal{S}^{(i+1)}\|^2$, and $\|\mathcal{S}^{(i+1)}\|^2$. If the eigenvalues of $\mathcal{S}$ have a Gaussian distribution, we know from Eq. (7) that $\|\mathcal{S}^{(i+1)}\|^2 = \sqrt{3} \|\mathcal{S}\|^2$ and $\|\mathcal{S}^{(i+1)}\|^2 = \sqrt{15} \|\mathcal{S}\|^2$. Another possibility is that the energy levels might be spaced so that $\mathcal{S}$ has only two transitions which are nearly resonant, forming an effective three-level system, and $\mathcal{S}$ contains a nonzero matrix element for only one of these transitions. In this case $N_0 = 3$, and if $\mathcal{S}$ is traceless, we expect $\|\mathcal{S}\|^2 = (3/2)(\alpha^{1-1/2} \|\mathcal{S}\|^2)$. In both cases $\|\mathcal{S}^{(i+1)}\|^2$ does not grow faster than $\|\mathcal{S}\|^2$ very rapidly. However, if $K > 1$, so that only a very small fraction of the matrix elements of $\mathcal{S}$ are selected by $\mathcal{S}$, we may have $\|\mathcal{S}^{(i+1)}\|^2 \sim (N_0)^{1-1/2} \|\mathcal{S}\|^2$; this case and the case of a Gaussian distribution will be discussed in the next section in connection with multiple-quantum NMR.

The factor $(2n)^{1/2} e^{\gamma/\phi}$ makes $S$ grow very rapidly as $j$ is increased, and fairly small values of $j$ still give very selective sequences. For example, if a Gaussian distribution of eigenvalues is assumed for $\mathcal{S}$, the selectivity of a third-order $10k$-quantum selective sequence with $\alpha = 1$, $K = \alpha$, and $\rho(n) = 0.028$ (from Table III) is $S = 1170$, a typical selected matrix element is more than three orders of magnitude larger than a typical nonselected matrix element, even near the limit of convergence. When $\|\mathcal{S}^{(i+1)}\|^2 \lesssim F(n)^{-1}$, $S$ will be much larger; in general, if $\|\mathcal{S}^{(i+1)}\|^2$ is scaled down by a factor of $A$, $\|\mathcal{S}^{(i+1)}\|^2$ is scaled down by a factor of $A^{-1}$, and $S$ increases by a factor of $A^{-1}$. We conclude that for many systems the use of cycles with only a finite order of selectivity is entirely satisfactory.

The effects of timing errors and imperfect phase shifts are more serious. Suppose that the length of subcycle $i$ is $\Delta t_i + \delta_i$, and that the phase is $\phi_i + \epsilon_i$, where $\sum_i \delta_i = \sum_i \epsilon_i = 0$ (Fig. 9). Then

$$S_{\epsilon} = \Delta t_i + \delta_i,$$

(42)

which is no longer purely $nk$-quantum selective; the matrix element for an $m$-quantum transition is multiplied by

$$\frac{1}{2n\Delta t} \sum_j \left( \Delta t_j + \delta_j \right) \exp \left[ \kappa l_x (\phi_j + \epsilon_j) \right],$$

(43)

instead of 0. Assuming $\delta_j \ll \Delta t_j$ and $\epsilon_j \ll 1$, we expand this out:

$$\frac{1}{2n\Delta t} \sum_j \left( \Delta t_j + \delta_j \right) \exp \left[ \kappa l_x (\phi_j + \epsilon_j) \right]$$

$$= \frac{1}{2n\Delta t} \left[ \left( \sum \Delta t_j \exp (\kappa l_x \phi_j) \right) + \sum \delta_j \exp (\kappa l_x \phi_j) \right]$$

$$+ \left( \sum \Delta t_j \exp (\kappa l_x \epsilon_j) \right) \exp (\kappa l_x \phi_j) \ldots \right].$$

(44)

The first term on the rhs corresponds to an ideal sequence and vanishes if $m$ is not a multiple of $n$. If $\epsilon_i$ and $\delta_i$ are uncorrelated with $\phi_i$, the last two terms reduce to $2n \left( \sum \Delta t_j \exp (\kappa l_x \phi_j) \right)$ and the number of subcycles increases (for example, by going to a higher-order selective sequence), this term decreases, so that the ratio $\|\mathcal{S}\|^{\text{selective}} / \|\mathcal{S}\|^{\text{max}}$ can be made arbitrarily large. However, if $\epsilon_i$ or $\delta_i$ are completely correlated with $\phi_i$ (so that, for example, every time the phase should be $\phi = 0$ it is actually $\phi = \phi_0$, $\|\mathcal{S}\|^{\text{selective}} / \|\mathcal{S}\|^{\text{max}}$ is not reduced by increasing the order of the sequence. Such a situation arises with a maliciously phase shifting device or for one in which a digital approximation is made to the exact setting. One way to reduce this error is to use two (or more) phase shifting devices, so that the total error is not well correlated with the total phase.

B. Application of nonideal selective sequences to multiple-quantum NMR

As mentioned earlier, an ideal $N$-quantum sequence that had no zero-quantum matrix elements could enhance the single $N$-quantum transition of an $N$-spin system by a factor of $N^2$. A nonideal $nk$-quantum selective sequence (which has zero-quantum matrix elements) will not work as well, for two reasons. First, there may be nonzero matrix elements for the populations of the states $m = -N/2$ (populations may be thought of as a special type of zero-quantum coherence, with the initial state identical to the final state). In this case, the effective Hamiltonian for the two levels will be $a \lambda^{1+} + a \lambda^{1-} + a \lambda^{1+} + a \lambda^{1-}$ (Fig. 10) instead of containing only $\lambda^{1+}$ and $\lambda^{1-}$. Depending on the relative size of the coefficient $a_1$, it may be impossible to transfer population completely into coherence. Statistically, we expect the coefficients to be of comparable size and in that case much of the population can be transferred into coherence.

A much more serious effect comes from the requirement that the average Hamiltonian expansion converge. In general, this would imply $\|\mathcal{S}^{(i+1)}\|^2 \ll F(n)^{-1}$. For the sequences described, the distribution of eigenvalues...
I

FIG. 10. Schematic illustration of one effect of zero-quantum coherences on an N-quantum selective sequence. The sequence causes the population difference between the states \( m = \pm \frac{N}{2} \) to be rotated about an arbitrary axis, and complete transfer of population difference into coherence may be impossible. Compare this to Fig. 8.

of \( \mathcal{C}_p \) is expected to resemble that of the eigenvalues of \( \mathcal{C}_{p,ss} \), which will be Gaussian if \( N \) is large. It is reasonable to assume that the eigenvalues of \( \mathcal{C}^{(0)} \) have a Gaussian distribution, since \( \mathcal{C}^{(0)} \) has many allowed zero-quantum transitions, and since the limit of \( \mathcal{C}^{(0)} \) as \( T \to 0 \) for Fig. 6(a) is \( \mathcal{C}_{p,ss} \). The convergence criterion of Eq. (41) can then be written as \( \| \mathcal{C}^{(0)} f_c \| \ll 3^{1/4} \mathcal{F}(\mu)^{1/8} \sim 2.1 (\mu^{1/4}) \).

From the definition of the norm of a \( 2^N \) by \( 2^N \) matrix we find

\[
\| \mathcal{C}^{(0)} f_c \| = \left( 2^N \sum_{m} \| \mathcal{C}^{(0)} f_c \|_m \right)^{1/2} = \sqrt{2^N} \sum_{m} \left( \mathcal{C}^{(0)} f_c \|_m \right)^{1/2}, \tag{45}
\]

where \( N_s \) is the number of possibly nonzero matrix elements, and the rms average only includes those elements. Therefore, to make an average excited matrix element comparable to 1 (which is needed if we want the effect of \( \mathcal{C}^{(0)} \) to approximate a selective 90° pulse between the levels \( m = \pm N/2 \) we need

\[
N_s \leq 2^N.
\]

There are \( 2^{2N} \) possibly nonvanishing matrix elements for a nonselective sequence, and if only zero-quantum and N-quantum elements are excited, \( N_s \sim 2^{2N} N^{-1/2} \).

For \( N=2 \), Eq. (46) cannot be satisfied. Assuming \( \| \mathcal{C}^{(0)} f_c \| \sim 1 \), we expect

\[
\langle \| \mathcal{C}^{(0)} f_c \|_m \rangle^{1/2} \sim 2^{N/2} N^{-1/2} \ll 1.
\]

This scales down the possible gain because the selective sequence effectively produces only a small rotation instead of a 90° pulse. In fact,

\[
\langle \| \mathcal{C}^{(0)} f_c \|_m \rangle^{1/2} \sim (N^{1/4}/2^{N/2}) (\mathcal{F}(\mu)^{1/8} / \mu^{1/8})
\]

so the gain is

\[
G' = \left[ \beta \sqrt{N/2} (\mathcal{F}(\mu)^{1/8} / \mu^{1/8}) \right] / \beta 2^{2N} \mathcal{F}(\mu)^{1/8} N^{3/4}
\]

This value of \( G' \) is only approximate, since it depends on the exact maximum permissible value of \( \| \mathcal{C}^{(0)} f_c \| \). The gain can be assigned to two effects. Only \( -1/\sqrt{N} \) as many transitions are being pumped so each one is \( \sqrt{N} \) times stronger; in addition, the N-quantum transition receives intensity from the equilibrium population difference of the extreme states for which the expectation value of \( I^1 \) is \( N^2 \), a factor of \( N \) greater than the expectation value averaged over all states. While this gain is large, a much larger gain is possible if \( N_s \) can be reduced.

C. Removal of 0-quantum operators from selective sequences

The gain can be increased if the zero-quantum coherences are removed from \( \mathcal{C}^{(0)} \). One way to do this is with the sequence shown in Fig. 11(a). The phase shift of \( \pi/N \) inverts the N-quantum coherence but leaves the zero-quantum coherence invariant, and the time re-
versal inverts every order of coherence, so the net result is that zero-quantum coherences are inverted every subsystem but N-quantum coherences are unaffected. The lowest-order average \( R^{(2)} \) for the sequence in Fig. 11(a) contains only \( N \)-quantum, \( 3N \)-quantum, ..., \((2k+1)N\)-quantum coherences after \( 2N \) subcycles. In an \( N \)-spin system, this makes \( N \ll 2^k \), condition (46) is satisfied, and the ideal gain \( G = N^2 \) of Table I becomes possible.

The easiest way to design such a sequence for \( S \) elements is shown in Fig. 11(b). \( S \gamma_p \), \( -S \gamma_p \), \( S \gamma_q \), and \( -S \gamma_q \) are all generated from the double-quantum sequence, mentioned earlier and illustrated in Fig. 11(c) with relative phases \( 0, \pi/2, \pi/4, \) and \( 3\pi/4 \), respectively. If \( \|S\gamma_p\|/\|S\gamma_q\| \ll 1 \), only the \( N \)-quantum transition appears. However, the value of \( K = \|S\gamma_p\|/\|S\gamma_q\| \) is now very large \((K \sim 2^k)\) so the selectivity \( S \) from Eq. (41) will be somewhat weaker, and should be reconsidered. High-order selective sequences with no zero-quantum contributions can be generated from Fig. 11(a) in exactly the same way that high-order selective sequences with zero-quantum contributions were generated from Fig. 4(a). A third-order 10-quantum selective sequence requires \((2N)^3 = 800 \) subcycles [instead of \((2N)^2 = 400 \) subcycles without suppression of zero quantum]. Equation (41) applies if \((2N)^{3/4} + 7/4 \) is replaced with \((2N)^{3/4} + 7/4 \) and \( N \) is replaced with \( 2N \), since each symmetrized phase cycling now requires \( 4N \) subcycles.

The assumption of a Gaussian distribution of eigenvalues is no longer valid, since \( R^{(2)} \) has only two non-zero matrix elements. Since \( R^{(2)} \) is Hermitian, the two elements have the same magnitude \( |R| \). The eigenvalues are \( \pm R, \) and \( 0 \) for all other states, because \( R^{(2)} \) is traceless. Therefore,

\[
\|R^{(2)}\| = (2^N(2)(2^N/4))^{1/2} = R^2(2^{N+1})^{1/2},
\]

\[
\|R^{(2)}\|^2 = (2^N(2)(2^N/16))^{1/2} = R^2(2^{N+1/2}) q R^{(2)} I^{(2)} \| R^{(2)} \|^2,
\]

\[
\|R^{(2)}\|^3 = (2^N(2)(2^N/64))^{1/2} = R^2(2^{N+1}) q R^{(2)} I^{(2)} \|^3.
\]

In the limit of convergence \((\|R^{(2)}\|^2 - F(n)^{-1})\) we find the calculated selectivity of a third-order \( 10 \)-quantum sequence with \( \alpha = 1, \) \( K = 2^{2k+1} \), and \( F(n) = 0.029 \) [see Eq. (41)] to be \( S = 0.025 \). However, we do not really need to have \( \|R^{(2)}\|^2 \) be this large; all we really want is \( R_I = \pi/2 \), so \( \|R^{(2)}\|^2 = 0.027 \), a factor of 1797 smaller. When \( \|R^{(2)}\|^2 \) has this value, we find \( S = 8.0 \times 10^4 \).

We can attain the maximum gain \( G \) if the \( 4(2^N - 2) \) non-\( N \)-quantum selective matrix elements that involve the \( m = \pm N/2 \) states only a small fraction of \( [(2N)^{1/2}] \) relative to the fraction transferred by the two \( N \)-quantum selective matrix elements. Since each selective matrix element is larger by a factor \( S \), the intensity of the selected transition [from Eq. (32)] is larger by a factor \( S^2 \), and we require

\[
2S^2 \gg 4(2^N - 2),
\]

\[
\|S\|^2 \gg 2^{k+1} - 4,
\]

which is satisfied in this example. We therefore conclude almost all the theoretical gain from an infinite-order selective sequence is attainable with a third-order selective sequence, and the potential gains in Table I should be approximately realizable.

Note that the theoretical maximum gain becomes difficult to achieve as \( N \) increases, for several reasons. From Eq. (53) we see that the required selectivity for maximum gain is proportional to \( (2^N)^{1/2} \), and therefore high-order selective sequences may be needed. However, the number of subcycles cannot be increased indefinitely, because each subcycle must have a minimum duration, and relaxation effects limit the maximum duration of the cycle to less than \( T_2 \). Typically, \( \|3C_{\Delta u}\| \sim 10 \) kHz and \( T_2 \sim 100 \) ms for liquid crystal systems, so no more than a few thousand subcycles would be possible; a third-order 18-quantum selective sequence that eliminates zero-quantum requires \((4N)^3 = 5184 \) subcycles, and therefore is impossible for many molecules. In addition, sample heating becomes a serious problem when many pulses are applied.

V. CONCLUSION

On the basis of the calculations presented in this paper and experimental results reported elsewhere, \( ^1 \) we believe that selective excitation will increase the number and size of molecules which can be studied by multiple-quantum NMR. We also expect that this approach to selective excitation of coherence and selective population inversion will be useful in a variety of systems.

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APPENDIX

In this Appendix we calculate the size of the first non-\( nk \)-quantum selective operator from a \( j \)-order \( nk \)-quantum selective sequence, which is \( (-J\gamma_p^{(j+1)})^{1/2} \).

To simplify the calculations we combine phase cycling and symmetrization into one operation, which turns a \( (j-2) \)-order \( nk \)-quantum selective subcycle (\( j \) odd) into a \( j \)-order \( nk \)-quantum selective cycle requiring \( 2n \) subcycles [Fig. 4(b)], assuming perfect phase shifts and no timing errors. This will allow us to write \( \beta^{(j+1)} \). As in our earlier calculations, we expand the propagator for the \( j \)-order \( nk \)-quantum selective sequence in powers of \( t_p \). The first nonselective term is \( \beta^{(j+1)} t_p \), which is proportional to \( \beta^{(j+1)} t_p \). All other terms proportional to \( t_p^{j+2} \) are \( nk \)-quantum selective, so the only possible non-term proportional to \( j^{(j+2)} \) is \( \beta^{(j+1)} t_p \). The product of the propagators for the subcycles has several terms proportional to \( \beta^{(j+1)} t_p \); they are...
Selecting excitation of multiple-quantum transitions

\[ \Pi \{ t_{c_{tl}} \} = -i \sum_{i} \left( \langle R_{(i+1)} \rangle_{t_{c_{t}}}^{\dagger} \right) t_{c_{tl}} \]  
(A1)

\[ - \sum_{i} \left( \langle R_{(i+1)} \rangle^{(t_{c_{t}})}_{c_{tl}} + \langle R_{(i)}^{(t_{c_{t}})} \rangle_{c_{tl}} \right) t_{c_{tl}} \]  
(A2)

\[ \frac{1}{2} \sum_{i} \left( \langle R_{(i)}^{(t_{c_{t}})} \rangle_{c_{tl}} + \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \right) \]  
(A3)

\[ - \sum_{i} \left( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} + \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \right) t_{c_{tl}} \]  
(A4)

\[ + \left[ i \sum_{i, j, k} t_{c_{t}}^{2} + \left( \frac{i}{2} \right) \sum_{i, j, k} t_{c_{t}}^{2} \right] \]  
(A5)

\[ \left( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} + \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \right) t_{c_{tl}} \]  
(A6)

\[ + \left( i \right) \sum_{i, j, k} t_{c_{t}}^{2} \left( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} + \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \right) \]  
(A7)

\[ + \left( i \right) \sum_{i, j, k} t_{c_{t}}^{2} \left( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} + \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \right) \]  
(A8)

\[ + \text{terms which are not proportional to } t_{c_{t}}^{2}. \]

We will assume that the subcycles are constructed by phase cycling and symmetrization, so that \( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} = 0 \) for all \( j \), and terms (A2)–(A5) vanish. In addition, terms (A1) and (A7) are unaffected by a phase shift of \( \phi = 2\pi/n \), so they are \( nk \)-quantum selective. Finally, term (A8) contains many \( nk \)-quantum selective portions; for example, \( \sum_{i, j, k} \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \) is \( nk \)-quantum selective if \( i \) and \( j \) are in the first half of the cycle and \( k \) is in the second half. As another example, if \( i = j = 1 \), the nns part of the summation \( \sum_{i} \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \), which contains \( (n-1) \) terms, is the same as a single term, since \( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} = 0 \). Straightforward but tedious counting of the remaining terms, assuming that their matrix elements add randomly, gives

\[ ||(R_{(i+1)}^{(t_{c_{t}})})_{c_{tl}}||^{2} = \frac{1}{n} \]

\[ F(n) = \frac{1}{n} \sum_{i} \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} ||^{2} t_{c_{t}}^{2} \]  
(A8)

The construction of the subcycles implies \( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} = \frac{1}{n} \), \( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} = \frac{1}{n} \), and therefore \( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \) is \( nk \)-quantum selective (note, however, the sum with \( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \)). We expect to retain convergence for the \( j \)-order selective sequence if \( ||(R_{(i+1)}^{(t_{c_{t}})})_{c_{tl}} || < F(n)^{-1} \).

As mentioned, we can use Eq. (A9) iteratively, to calculate the first nns term as a function of successively lower-order terms. To begin the iteration, however, we need an expression for \( ||(R_{(i+1)}^{(t_{c_{t}})})_{c_{tl}} || \) for a first-order selective sequence [Fig. 4(b)] with \( K_{e} \) nonselective. We have shown that [Eq. (16)]

\[ ||(R_{(i+1)}^{(t_{c_{t}})})_{c_{tl}} || = (i/2) \sum_{i, j, k} t_{c_{t}}^{2} \left( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} + \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \right) \]  
(A9)

\[ \left( i \right) \sum_{i, j, k} t_{c_{t}}^{2} \left( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} + \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \right) \]  
(A10)

Again a substantial fraction of the rhs cancels or is forced to be selective. Most of the remaining terms on the rhs cancel with terms such as \( \langle R_{(i+1)}^{(t_{c_{t}})} \rangle_{c_{tl}} \). The result is that, if \( ||(R_{(i+1)}^{(t_{c_{t}})})_{c_{tl}} || < F(n)^{-1} \), we expect \( ||(R_{(i+1)}^{(t_{c_{t}})})_{c_{tl}} || \) to be selective. In this case no further reduction is possible.
Let $||\mathcal{R}e_{(j)}||^2 = \alpha K ||\mathcal{R}e(0)||^2$ (see Sec. IV. A). We can now write

$$||\mathcal{R}e(0)\Delta \tau_p||_{\text{max}} \leq \frac{\alpha K^3}{12} ||(\mathcal{R}e(0)\Delta \tau_p)||^2 + \frac{n^2 \alpha^2 K^2}{8} ||(\mathcal{R}e(0)\Delta \tau_p)||^2 \left(2(\mathcal{R}e(0)\Delta \tau_p)||^2 ||\mathcal{R}e(0)\Delta \tau_p||^2 \right)^{1/2}.$$  

(A12)

We define the selectivity $S$ to be $K^{1/2}||\mathcal{R}e(0)||/||\mathcal{R}e(0)||_{\text{max}}$, which is the ratio of a typical nonzero matrix element of $\mathcal{R}e(0)$ to a typical nonzero matrix element of $(\mathcal{R}e(0))_{\text{max}}$. We would like to calculate $S$ as $t_c = (2n)^{-1/2} \Delta \tau_p$, which approaches the limit of convergence, which is $||\mathcal{R}e(0)\Delta \tau_p||^2 = F(n)^{-1}$. Since $t_c = (2n)^{-1/2} \Delta \tau_p$, $(\mathcal{R}e(0)\Delta \tau_p)^{-1} \approx (2n)^{-1/2} F(n)^{-1}$, and $S = K^{1/2}||\mathcal{R}e(0)||/||\mathcal{R}e(0)||_{\text{max}} = K^{-1} F(n)^{-1} (2n)^{1/2} F(n)^{1/2} \times \left[\frac{n^2 \alpha^2 K^2}{8} + \frac{n^2 \alpha^3}{12} \frac{F(n)^{-1}}{(\mathcal{R}e(0)\Delta \tau_p)||^2} \right]^{1/2}$.  

(A15)

Equation (A15) is the same as Eq. (41).

Let $t_c = (2n)^{-1/2} \Delta \tau_p$, we want the first-order sequence to converge. If $||\mathcal{R}e(0)\Delta \tau_p||^2 \ll 1$, we see that $||\mathcal{R}e(0)\Delta \tau_p||_{\text{max}} \ll 1$. Combining this with our earlier results, we expect a $j$-order $nk$-quantum selective sequence which is constructed from $\frac{1}{2}(j+1)$ phase cycles and $\frac{1}{2}(j+1)$ symmetrizations (and therefore has a cycle time $t_c = (2n)^{1/2} \Delta \tau_p$) to converge if

$$||\mathcal{R}e(0)\Delta \tau_p||^2 \ll 1 \text{ and } ||(\mathcal{R}e(0)\Delta \tau_p)||^2 \ll F(n)^{-1}.$$